Quaternionic Root Systems and Subgroups of the $Aut(F_4)$

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Cayley-Dickson doubling procedure is used to construct the root systems of some celebrated Lie algebras in terms of the integer elements of the division algebras of real numbers, complex numbers, quaternions and octonions. Starting with the roots and weights of $SU(2)$ expressed as the real numbers one can construct the root systems of the Lie algebras of $SO(4), SP(2) \approx SO(5), SO(8), SO(9), F_4$ and E_8 in terms of the discrete elements of the division algebras. The roots themselves display the group structures besides the octonionic roots of E_8 which form a closed octonion algebra. The automorphism group $Aut(F_4)$ of the Dynkin diagram of F_4 of order 2304, the largest crystallographic group in 4-dimensional Euclidean space, is realized as the direct product of two binary octahedral group of quaternions preserving the quaternionic root system of F_4. The Weyl groups of many Lie algebras, such as, $G_2, SO(7), SO(8), SO(9), SU(3)XSU(3)$ and $SP(3) \times SU(2)$ have been constructed as the subgroups of $Aut(F_4)$. We have also classified the other non-parabolic subgroups of $Aut(F_4)$ which are not Weyl groups. Two subgroups of orders192 with different conjugacy classes occur as maximal subgroups in the finite subgroups of the Lie group G_2 of orders 12096 and 1344 and proves to be useful in their constructions. The triality of $SO(8)$ manifesting itself as the cyclic symmetry of the quaternionic imaginary units e_1, e_2, e_3 is used to show that $SO(7)$ and $SO(9)$ can be embedded triply symmetric way in $SO(8)$ and F_4 respectively.

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INTRODUCTION

There are a few celebrated Lie algebras which seem to be playing important roles in understanding the underlying symmetries of the unified theory of all interactions. The most popular ones are the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 and the related groups [1]. The groups $Spin7$ and G_2 are proposed as the holonomy groups for the compactification of the M-theory from 11 to 4 dimensional space-time [2]. It is also well known that two orthogonal groups $SO(8)$ and $SO(9)$ are the little groups of the massles particles of string theories in 10-dimensions and the M-theory in 11-dimensions respectively. The fact that $SO(9)$ can be embedded in the exceptional Lie group F_4 in a triply symmetric way and the non-compact $F_{4(-25)}$ can be embedded in the Lorenz group $SO(25,1)$ indicates the importance of the exceptional group F_4 [3]. The largest exceptional group E_8 which had been suggested as the unified theory of the electroweak and strong interactions with three generations of lepton-quark families [4] naturally occurred as the gauge symmetry of the $E_8 \times E_8$ heterotic string theory [5]. It has many novel mathematical aspects [6] which has not been exploited in physics. It was known that a non-compact version of E_7 manifests itself as a global symmetry of the 11-dimensional supergravity [7]. Some of its maximal subgroups show themselves as local symmetries [8]. The E_6 has been suggested as a unified theory of electroweak and strong interactions [9].

The Weyl groups of these groups are also important for the invariants of the Lie groups can be deduced using the related Weyl groups. The Weyl groups of the exceptional Lie groups F_4, E_6, E_7 and E_8 correspond to some finite subgroups of the Lie groups of $O(4), O(6), O(7)$ and $O(8)$ respectively [10].

It has been shown in some details that the finite subgroups of $O(4)$ can be classified as direct products of finite subgroups of quaternions [11], isomorphic to the finite subgroups of $SU(2)$, which is the double cover of $SO(3)$. Therefore the relevant Weyl groups of the Lie groups $F_4, SO(9)$ and $SO(8)$ correspond to some finite subgroups of $O(4)$ [12]. Similarly, the Weyl groups of some rank-3 Lie algebras can be obtained from the finite subgroups of $O(3)$. Interestingly enough, the relevant root systems can be represented as discrete quaternions and the Weyl groups can be realized as the left-right actions of the quaternions on the root systems. When one considers the finite subgroups of $O(8)$ it is natural to encounter with the discrete octonions which represent the root system of E_8 where the root system of E_7 is described by imaginary octonions [13]. The automorphism group of octonionionic root system of E_7 turns out to be a finite subgroup of G_2 of order 12096 [14]. In what follows we will restrict ourselves to the quaternionic root system of $F_4, SO(9), SO(8), SO(7), SP(3)$ and construct explicitly their Weyl groups as finite subgroups of $O(4)$ .The largest group of interest here is the $Aut(F_4)$ of order $2304 = 48 \times 48$ which is the direct product of the binary

octahedral group with itself [15]. We follow a chain of decomposition of $Aut(F_4)$ into its relevant subgroups, some of which, are maximal subgroups in the finite subgroups of G_2 of orders 12096 and 1344 [16].

The paper is organized as follows. In section 2 we start with the scaled roots $\pm 1, 0$ and the weights $\pm\frac{1}{2}$ of $SU(2)$ and using the Cayley-Dickson doubling procedure we construct the roots of $SO(4)$ and $SP(2) \approx SO(5)$ in terms of complex numbers. Further doubling of the roots of $SP(2) \approx SO(5)$ leads to the quaternionic roots of $SO(8)$ [13, 16]. The 8-dimensional vector and spinor representations of Spin-8 constitute the short roots of F_4 . Doubling of two sets of quaternionic roots of F_4 leads to the octonionic roots of E_8 . The triality of $SO(8)$ is then coded in the cyclic symmetry of the quaternionic imaginary units. In section 3 we introduce the finite subgroups of $SU(2)$ in terms of quaternions and explain their geometric properties. We explain how to construct the $Aut(F_4)$ and the Weyl groups of $F_4, SO(9)$ and $SO(8)$ [15]. In section 4 we construct the root systems of $SO(7)$ and G_2 by folding the Coxeter-Dynkin diagram of $SO(8)$ [17] which displays the three-fold embeddings of $SO(7)$ into $SO(8)$ .The Weyl groups of $SO(7)$, G_2 and $SP(3)$ are constructed in terms of quaternions. In section 5 we discuss the subgroup chains of $Aut(F_4)$ and find out the explicit expressions of the groups down to the groups of order 192. A particular emphasis is given to two groups of orders 192 since they appear as the maximal subgroups in the finite subgroups of G_2 of orders 12096 and 1344. Finally in section 6 we further elaborate the geometric aspects of the symmetries discussed in the previous sections.

ROOT SYSTEMS WITH THE CAYLEY-DICKSON DOUBLING PROCEDURE

The Cayley- Dickson doubling is a procedure to build the elements of division algebras starting with the real numbers. Let us denote by p, q, r, s the elements of a division algebra other than the octonions. Then the pairs (p, q) and (r, s) with the multiplication rule

$$(p,q)(r,s) = (pr - s\bar{q}, rq + \bar{p}s) \tag{1}$$

constitute the elements of a division algebra in higher dimension. The celebrated Hurwitz's theorem[18] states that there are only four division algebras, namely, real numbers, complex numbers, quaternions and octonions. Starting with the complex numbers at every higher level of division algebras one introduces one complex number, say, e_1, e_2 and e_7 which anti-commute with each other and satisfy the relation $e_1^2 = e_2^2 = e_7^2 = -1$. Doubling of the real numbers constitutes the complex numbers, two sets of complex numbers define the quaternions and finally a pair of quaternions defines the octonions under the definition (1). Let us revise the work of reference [13] by starting with the roots $\pm 1, 0$ and the weights $\pm\frac{1}{2}$ of $SU(2)$. A pair of set $\pm 1, 0$ leads to the roots of $SO(4)$:

$$(\pm 1, 0) = \pm 1, (0, \pm 1) = \pm e_1 (\text{we} \quad \text{use} \quad e_1 \quad \text{for} \quad \text{the} \quad \text{imaginary} \quad \text{number} \quad i) \tag{2}$$

The non-zero roots $\pm 1, \pm e_1$ of $SO(4)$ form a cyclic group of order 4. The weights of the spinor representation $(\underline{2}, \underline{2})$ of $SO(4)$ can be taken as

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right) = \frac{1}{2}\left(\pm 1 \pm e_1\right). \tag{3}$$

The roots in (2) and the weights in (3) constitute the scaled roots of $SP(2) \approx SO(5)$,

$$SP(2) \approx SO(5) : \pm 1, \pm e_1, \frac{1}{2}(\pm 1 \pm e_1). \tag{4}$$

When the short roots are scaled to the unit norm then the roots of $SP(2)$ form a cyclic group of order 8. A non-trivial structure will arise when two sets of the roots of $SP(2)$ are paired as $(SP(2), SP(2))$ where the long roots match with the zero roots while the short roots match with the short roots leading to the quaternionic roots of $SO(8)$:

$$T : \left\{ \pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \right\} \tag{5}$$

where we have used $e_3 e_1 = -e_1 e_3 = e_2$. If we include the pairing of the short roots with the zero roots we obtain

$$V_1' : \left(\frac{1}{2}(\pm 1 \pm e_1), 0\right) = \frac{1}{2}(\pm 1 \pm e_1), \left(0, \frac{1}{2}(\pm 1 \pm e_1)\right) = \frac{1}{2}(\pm e_2 \pm e_3) \tag{6}$$

These are the weights of the 8-dimensional representation of $SO(8)$ and together with the roots in (5) they represent the roots of $SO(9)$. The cyclic symmetry of the quaternionic imaginary units would lead to the weights of the two 8-dimensional spinor representations of $SO(8)$ which represent the weights of the 16-dimensional spinor representation of $SO(9)$:

$$
\begin{array}{cc}
& \frac{1}{2}(\pm 1 \pm e_2) \quad \frac{1}{2}(\pm 1 \pm e_3) \\
V_2' : & \frac{1}{2}(\pm e_3 \pm e_1) \quad V_3' : \frac{1}{2}(\pm e_1 \pm e_2)
\end{array}
\tag{7}
$$

The set of quaternions in (5-7) constitutes the scaled roots of F_4 . A further doubling the set of roots of F_4 will lead to the octonionic roots of E_8 [13]:

$$
\begin{aligned}
(T,0) = & \quad T, \quad (0,T) = e_7 T \\
(V_1', V_1') = & \quad V_1' + e_7 V_1' \\
(V_2', V_3') = & \quad V_2' + e_7 V_3' \\
(V_3', V_2') = & \quad V_3' + e_7 V_2'
\end{aligned}
\tag{8}
$$

where one can define $e_4 = e_7 e_1$, $e_5 = e_7 e_2$, $e_6 = e_7 e_3$ .We note that when the roots in (6-7) are multiplied by $\sqrt{2}$ to make the norm 1 then the 48 set of quaternions are the elements of the binary octahedral group O of $SU(2)$ where T represents the binary tetrahedral subgroup of order 24.

BINARY OCTAHEDRAL GROUP AND THE $Aut(F_4)$

Some of the material of this section have been discussed in reference [12]. The finite subgroups of $SO(3)$ are well known: icosahedral group of order 60, octahedral group of order 24, tetrahedral group of order 12, and dihedral and cyclic groups of various orders [19]. Their double covers are the finite subgroups of quaternions which are related to the ADE series of the Lie algebras through the McKay correspondence [20]. Our interest here solely are constrained to the binary octahedral group whose direct product with itself is isomorphic to the $Aut(F_4)$ which can be realized as the left and right actions of the quaternionic elements on the quaternionic roots of F_4. The root system of F_4 has very interesting geometrical structures which has not been discussed in the literature. We classify the elements of the binary octahedral group as sets of the hyperoctahedra in 4-dimensions [21]:

$$
\begin{aligned}
& V_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\} \\
T : \quad & V_+ = \left\{\frac{1}{2} \pm 1 \pm e_1 \pm e_2 \pm e_3\right\}, \qquad \text{even number of } (+) \text{ signs} \\
& V_- = \overline{V_+} = \left\{\frac{1}{2} \pm 1 \pm e_1 \pm e_2 \pm e_3\right\}, \quad \text{even number of } (+) \text{ signs}
\end{aligned}
\tag{9}
$$

where $\overline{V_+}$ is the quaternionic conjugate of V_+.

$$
\begin{aligned}
& V_1 = \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\right\} \\
T' : \quad & \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\right\} \\
& \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\right\}
\end{aligned}
\tag{10}
$$

$$
O : \quad T \oplus T'
\tag{11}
$$

Here each of V_0 , V_+ and V_- represents the vertices of a hyperoctahedron in 4-dimensions and any two hyperoctahedra form a hypercube in 4-dimensions with 16 vertices. The set of quaternions T in (9a) not only constitute the non-zero roots of $SO(8)$ but also represent a polytope $3, 4, 3$ called 24-cell [21]. The set of quaternions in T' are the duals of T ; consequently any $V_i (i = 1, 2, 3)$ is a hyperoctahedron and any two hyperoctahedra form the vertices of a hypercube. We give the multiplication table of these sets of quaternions in Table1 to understand the structure of the binary octahedral group. Here V_0 is the quaternion group and form an invariant subgroup both in T and O.

A general element of $O(4) \approx SU(2) \times SU(2)$ can be defined as follows. Denote by p, q the quaternions of unit norm acting on an arbitrary quaternion $r = r_0 + r_1 e_1 + r_2 e_2 + r_3 e_3$

$$
{}^{[29]}r \;\rightarrow\; prq : [p,q]
\tag{12}
$$

$$
r \;\rightarrow\; p\bar{r}q : [p,q]^*
\tag{13}
$$

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

TABLE I: Multiplication table of the binary octahedral group

where $\bar{r}$ is the quaternion conjugate $\bar{r} = r_0 - r_1 e_1 - r_2 e_2 - r_3 e_3$. For arbitrary quaternions p, q with unit norm the elements $[p, q]$ and $[p, q]^*$ form a six parameter group leaving the norm $r\bar{r} = \bar{r}r$ invariant. When written in terms of matrices the group elements $[p, q]$ and $[p, q]^*$ have determinants $+1$ and -1 respectively. Therefore the elements $[p, q]$ form a subgroup $SO(4) \approx \frac{SU(2)xSU(2)}{Z_2}$ of $O(4)$. In the reference [12] we have proven that the Weyl group $W(F_4)$ can be compactly written as the union of elements,

$$W(F_4) = [T, T] \oplus [T\prime, T\prime] \oplus [T, T]^* \oplus [T\prime, T\prime]^* \tag{14}$$

The automorphism group $Aut(F_4)$ is the semi-direct product of the Weyl group $W(F_4)$ with the Z_2 symmetry of the Coxeter-Dynkin diagram of F_4,

$$^{[30]}Aut(F_4) \equiv (O, O) \oplus (O, O)^* \approx W(F_4) : Z_2. \tag{15}$$

The generators of $Aut(F_4)$ can be obtained from the Coxeter-Dynkin diagram of F_4 where the simple roots are given in terms of scaled quaternions.
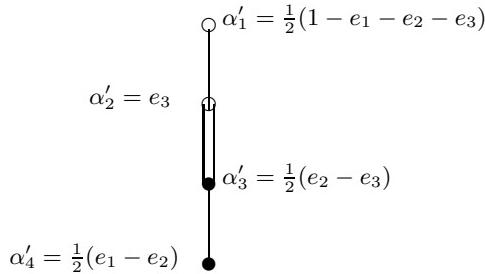
FIG. 1: The Coxeter-Dynkin diagram of F_4.

The regular simple roots α_i are related to α'_i by $\alpha_i = \sqrt{2}\alpha'_i (i = 1, 2, 3, 4)$. The $Aut(F_4)$ is generated by the elements

$$[\alpha'_1, -\alpha'_1]^*, [\alpha'_2, -\alpha'_2]^*, [\alpha'_3, -\alpha'_3]^*, [\alpha'_4, -\alpha'_4]^*, [\frac{1}{\sqrt{2}}(e_2 + e_3), -e_2] \tag{16}$$

The first four generators in (16) represent the reflections in the roots $\alpha_i (i = 1, 2, 3, 4)$ and generate the Weyl group $W(F_4)$ and the last term stands for the diagram symmetry of F_4 which transforms, by conjugation, $\alpha'_1 \leftrightarrow \alpha'_4$ and $\alpha'_2 \leftrightarrow \alpha'_3$. An extended Coxeter-Dynkin diagram of F_4 can be used to obtain the Coxeter-Dynkin diagrams of its maximal Lie algebras. We will discuss all starting with $SO(9)$.

The Parabolic Subgroups of F_4

$SO(9)$

We have shown in reference [12] that the Weyl group $W(SO(9))$ can be represented by the set of group elements

$$[V_0, V_0], [V_+, V_+], [V_-, V_-], [V_0, V_0]^*, [V_+, V_+]^*, [V_-, V_-]^* \tag{17}$$

$$[V_1, V_1], [V_2, V_3], [V_3, V_2], [V_1, V_1]^*, [V_2, V_3]^*, [V_3, V_2]^* \tag{18}$$

This is a group of order 384. The $W(SO(9))$ can be embedded in the $W(F_4)$ triply symmetric way by permuting the quaternionic imaginary units e_1, e_2, e_3 in the cyclic order. It is an inner automorphism of $W(F_4)$ which replaces the elements in (18) by the corresponding elements where the indices1,2,3 are permuted in the cyclic order. This permutation of the indices leaves the set of elements in (17) invariant as expected. Actually the set of elements in (17) constitute the elements of the Weyl group $W(SO(8))$. The Weyl group $W(SO(9))$ has a very interesting geometrical aspect; it is the largest symmetry preserving the 4-dimensional hyperoctahedron. One can show that the group elements in (17-18) leave the set of elements in V_1 invariant which is one of those 6 hyperoctahedra of the 48 roots of F_4. This has to be expected anyway because the set of roots $V_1/\sqrt{2}$ are the short roots of $SO(9)$ and has to be rotated to each other by the elements of $W(SO(9))$. Since the weights of the 16-dimensional spinor representation are represented by the quaternions $\frac{1}{\sqrt{2}}(V_2 + V_3)$ corresponding to the vertices of a cube in 4-dimensions they are also preserved by the elements of $W(SO(9))$ in (14a-b). Embedding $W(SO(9))$ in $Aut(F_4)$ can be made with a six fold cyclic symmetry under the conjugation, say, by $[V_+, V_1]W(SO(9))[V_-, V_1]$ where $[V_+, V_1]^6 = [V_0, V_0]$. This leads to six conjugate representations of $W(SO(9))$ in $Aut(F_4)$ in each of which one of the six hyperoctahedra $V_0, V_{\pm}, V_i(i = 1, 2, 3)$ is left invariant by $W(SO(9))$. There are other subgroups of $Aut(F_4)$ of order 384 not isomorphic to the Weyl group $W(SO(9))$. We will discuss them in section 5. Now we discuss the Weyl group of the maximal subalgebra $SU(2) \times SP(3)$ of F_4.

$$SU(2) \times SP(3)$$

The algebra $SU(2) \times SP(3)$ can be represented by the Coxeter-Dynkin diagram shown in figure 2.
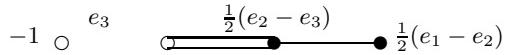
FIG. 2: The Coxeter-Dynkin diagram of $SU(2) \times SP(3)$.

The reflection generators on the simple roots are represented by

$$r_0 = [1, -1]^*, r_1 = [e_3, -e_3]^*, r_2 = [\frac{1}{\sqrt{2}}(e_2 - e_3), -\frac{1}{\sqrt{2}}(e_2 - e_3)]^*, r_3 = [\frac{1}{\sqrt{2}}(e_1 - e_2), -\frac{1}{\sqrt{2}}(e_1 - e_2)]^* \quad (19)$$

and will generate the set of roots

$$\begin{array}{cc} SU(2) & SP(3) \\ \pm 1 & \pm e_1, \pm e_2, \pm e_3 \\ & \frac{1}{2}(\pm e_1 \pm e_2) \\ & \frac{1}{2}(\pm e_2 \pm e_3) \\ & \frac{1}{2}(\pm e_3 \pm e_1) \end{array} \quad (20)$$

The long roots $\pm e_1, \pm e_2, \pm e_3$ of $SP(3)$ form the vertices of an octahedron. Therefore the Weyl group $W(SP(3))$ is the symmetry of the octahedron in 3-dimension. Since the product of two reflections is a rotation around some axis the proper rotation subgroup of $W(SP(3))$ is generated by

$$R = r_1 r_2 = [\frac{1}{\sqrt{2}}(1 - e_1), \frac{1}{\sqrt{2}}(1 + e_1)], S = r_2 r_3 = [\bar{t}, t] \quad (21)$$

with $t = \frac{1}{2}(1 + e_1 + e_2 + e_3)$. Here the generators satisfy the generation relations of an octahedral group [23]

$$R^4 = S^3 = (RS)^2 = [1, 1]. \quad (22)$$

It is one of the finite subgroups of $SO(3)$ isomorphic to the symmetric group S_4. Another generator $(r_1 r_2 r_3)^3 = [1, 1]^*$ commutes with the generators R and S so that the maximal group of the $SP(3)$ roots is the group $W(SP(3)) \approx S_4 \times Z_2$, a group of order 48. The Z_2 group of $W(SU(2))$ is generated by $[1, -1]^*$ which commutes with the generators of $W(SP(3))$. Therefore the Weyl group $W(SU(2))xW(SP(3))$ is isomorphic to the group $S_4 \times Z_2^2$ of order 96. The group elements are represented by the pair of quaternions

$$[p, \pm\bar{p}], [p', \pm\bar{p}], [p, \pm\bar{p}]^*, [p', \pm\bar{p}]; p \in T, p' \in T'. \tag{23}$$

Since the vertices of the octahedron are represented by the imaginary quaternions $\pm e_1, \pm e_2, \pm e_3$ one can naturally ask the question: what is the maximal group which preserves the quaternion algebra of the set of quaternions $\pm e_1, \pm e_2, \pm e_3$? It is well known that when p is the unit quaternion with non-zero real component then the transformation $e\prime_i = p e_i \bar{p}$ is the only transformation which preserves the quaternion algebra and is isomorphic to the group $SO(3)$. This implies that the finite subgroup of $SO(3)$ which preserves the set of quaternions $\pm e_1, \pm e_2, \pm e_3$ is the octahedral group represented by the elements $[p, \bar{p}], [p\prime, \bar{p}]$ which is isomorphic to the symmetric group S_4.

$$SU(3) \times SU(3)$$

From the extended Coxeter-Dynkin diagram of F_4 we obtain that the Coxeter-Dynkin diagram of $SU(3)xSU(3)$:

FIG. 3: The Coxeter-Dynkin diagram of $SU(3) \times SU(3)$. Here $\bar{t} = \frac{1}{2}(1 - e_1 - e_2 - e_3)$, $s\prime_1 = \frac{1}{2}(e_1 - e_2)$ and $s\prime_2 = \frac{1}{2}(e_2 - e_3)$.

Note that one of the $SU(3)$ is represented by the short roots. The non zero roots of $SU(3)xSU(3)$ are given by

$$\pm 1, \pm t, \pm \bar{t}, \pm s_1', \pm s_2', s_3' \tag{24}$$

where $s_3' = \frac{1}{2}(e_3 - e_1)$ .Using the standard technique one can form the elements of $W(SU(3)) \times W(SU(3))$ of order 36 which is the direct product of two symmetric group S_3. A further symmetry is the diagram automorphism of $SU(3) \times SU(3)$ which can be made by an element $c = [1, \frac{1}{\sqrt{2}}(e_1 - e_2)]$ which permutes the simple roots and preserve the Cartan matrix of the algebra $SU(3)xSU(3)$. An extension of the Weyl group $W(SU(3)) \times W(SU(3))$ by the element $c = [1, \frac{1}{\sqrt{2}}(e_1 - e_2)]$ leads to, up to conjugation, the group $Aut(SU(3)xSU(3)) \approx [W(SU(3)) \times W(SU(3))] : Z_4$ [24] where Z_4 is the cyclic group of order 4 generated by the element c. The set of elements can be represented by $[p, q] \oplus [p, q]^*$ where p, q take arbitrary values from the set of scaled roots $p, q \in \{\pm 1, \pm t, \pm \bar{t}, \pm s_1, \pm s_2, \pm s_3\}$ where $s_i = \sqrt{2}s\prime_i (i = 1, 2, 3)$.

$SO(8)$ AND ITS SUBGROUPS

The $SO(8)$ algebra plays a special role when embedding in F_4 since the long roots of F_4 are the roots of $SO(8)$. Its Coxeter-Dynkin diagram illustrates the triality in terms of the cyclic symmetry of the quaternionic imaginary units:

The Weyl group $W(SO(8))$ is represented by the set of elements (17) and the $Aut(S(O(8))$ is isomorphic to the Weyl group $W(F_4)$. Since in reference [12] we have worked $SO(8)$ in some detail here we will deal with its two special subgroups $SO(7)$ and G_2.
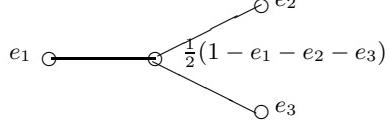
FIG. 4: The Coxeter-Dynkin diagram of $SO(8)$.

$$SO(7)$$

The $SO(7)$ diagram can be obtained from that of $SO(8)$ by folding two branches and averaging the corresponding simple roots [17].
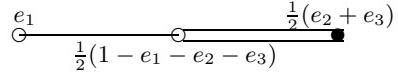

FIG. 5: The Coxeter-Dynkin diagram of $SO(7)$.

Figure 5. The Coxeter-Dynkin diagram of $SO(7)$.

Denote by the reflection generators r_1, r_2, r_3 of $SO(7)$ corresponding to the simple roots e_1, $\bar{t} = \frac{1}{2}(1 - e_1 - e_2 - e_3)$, $\frac{1}{2}(e_2 + e_3)$ respectively which can be expressed as

$$r_1 = [e_1, -e_1]^*, r_2 = [\bar{t}, -\bar{t}]^*, r_3 = [\frac{1}{\sqrt{2}}(e_2 + e_3), -\frac{1}{\sqrt{2}}(e_2 + e_3)]^* \tag{25}$$

One can also express r_3 in terms of the simple roots of $SO(8)$ as the product of reflection generators corresponding to e_2 and e_3 rather than the one in (25). That would give us $r_3 = [e_2, -e_2]^*[e_3, -e_3]^* = [e_1, -e_1]$ which gives the same result when acting on the roots of $SO(7)$. If we define $d_1 = \frac{1}{\sqrt{2}}(e_2 - e_3)$ then we can write the $W(SO(7))$ generators as

$$r_1 = [e_1, -d_1\bar{e}_1\bar{d}_1]^*, r_2 = [\bar{t}, -d_1 t \bar{d}_1]^*, r_1 = [e_1, -d_1\bar{e}_1\bar{d}_1] \tag{26}$$

The generators $a = r_1 r_2$ and $b = (r_1 r_2 r_3)^2$ satisfy the generation relation

$$a^3 = b^3 = (ab)^2 = [1, 1] \tag{27}$$

which is the generation relation of the tedrahedral group of order 12 isomorphic to the group A_4 of the even permutations of four letters [23]. The group elements can be written as

$$[p, d_1 \bar{p} \bar{d}_1], p \in T. \tag{28}$$

One can check that the elements

$$[1, -1], [1, \pm 1]^* \tag{29}$$

preserve the simple roots by conjugation. This means that the tetrahedral group in (28) can be extended by the elements in (29) so that the whole set of elements will read

$$[p, \pm d_1 \bar{p} \bar{d}_1], [p, \pm d_1 \bar{p} \bar{d}_1]^*. \tag{30}$$

We note that the set of elements

$$[p, d_1 \bar{p} \bar{d}_1], [p, d_1 \bar{p} \bar{d}_1]^* \tag{31}$$

form a group isomorphic to the octahedral group S_4 . The element $[1, -1]$ commutes with the elements of S_4 in (31). Therefore the set of elements represent a group isomorphic to the group $S_4 \times Z_2$ which is the Weyl group $W(SO(7))$ of order 48. This is the group isomorphic to $W(SP(3))$ represented by (??-??). We could have different foldings of $SO(8)$ diagram other than the one shown in figure 5. This would lead to replacing the quaternion d_1 in (30) by $d_2 = \frac{1}{2}(e_3 - e_1)$ and $d_3 = \frac{1}{2}(e_1 - e_2)$. By replacing d_1 by d_2 and d_3 in (30) we obtain three different embeddings of $SO(7)$ in $SO(8)$. When we stick to the representation of $W(SO(7))$ in (30) we can show that the 24 non-zero roots of $SO(8)$ can be decomposed as

$$\pm 1, \pm e_1, e_2, -e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm (e_2 + e_3)), \frac{1}{2}(\pm 1 \pm e_1 + (e_2 - e_3)) \tag{32}$$

which represent 18 non-zero roots of $SO(7)$ and the remaining ones are the 6 non-zero weights of the 7-dimensional representation of $SO(7)$

$$-e_2, e_3, \frac{1}{2}(\pm 1 \pm e_1 - (e_2 - e_3)). \tag{33}$$

Three different embeddings of $SO(7)$ in $SO(8)$ can be realized by permuting the indices $(1, 2, 3)$ in (32-33)in the cyclic order.

$$G_2$$

The Coxeter –Dynkin diagram of G_2 can be obtained from that of $SO(8)$ by folding three branches and taking the average of the outer simple roots [17].

$$\frac{1}{2}(1 - e_1 - e_2 - e_3) \qquad \Longrightarrow \qquad \frac{1}{3}(e_1 + e_2 + e_3)$$
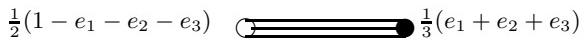
FIG. 6: The Coxeter-Dynkin diagram of G_2.

Let us denote by $I = \frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$ with $I^2 = -1$. The simple roots scaled by $\sqrt{2}$ are given by $\alpha_1 = \frac{1}{2}(1 - \sqrt{3}I) = e^{-\frac{\pi}{3}I}$, $\alpha_2 = \frac{I}{\sqrt{3}}$ and the reflection generators read $r_1 = [e^{-\frac{\pi}{3}I}, -e^{-\frac{\pi I}{3}}]^*$, $r_2 = [I, -I]^*$. The group $W(G_2)$ generated by r_1 and r_2 is the dihedral group D_6 of order 12. One can obtain the 12 non-zero roots of G_2 by acting the generators r_1 and r_2 on the simple roots. The weights of the 7-dimensional representation can be obtained from the highest weight $\sqrt{\frac{2}{3}}e^{\frac{\pi}{6}I}$.

A remark is in order. We can summarize the discussion in this section that the $SO(7)$ can be embedded in $SO(8)$ triply symmetric way and the G_2 takes place in the intersection of these three $SO(7)$ in $SO(8)$ [25]
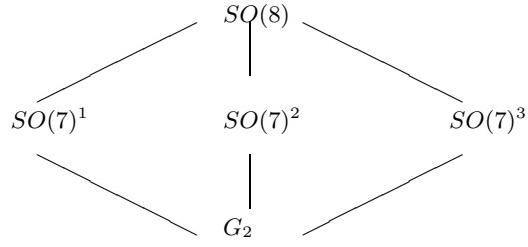
FIG. 7: Three-fold embedding of $SO(7)$ in $SO(8)$.

NON-PARABOLIC SUBGROUPS OF $Aut(F_4)$

So far we have discussed the parabolic subgroups of $Aut(F_4)$ related with the Lie sub-algebras of F_4 . As we have mentioned before the $Aut(F_4)$ is the largest crystallographic group in 4-dimensions and deserves further analysis regarding its chain decomposition through its maximal subgroups which could be useful for the crystallography in 4- dimensions. First we discuss the maximal subgroups of $Aut(F_4)$. We will give the group elements in terms of quaternions and distinguish the groups by their orders and conjugacy classes. The group orders and conjugacy classes are not sufficient to understand the group structures. Since we will write down the group elements explicitly in terms of quaternions the distinguishing the groups of the same order will not create a problem. Nevertheless we will denote a group of interest with its order together with its conjugacy classes in a parenthesis and display the group elements in terms of quaternions. For example, the group $Aut(F_4)$, being of order 2304 with 29 conjugacy classes will be shortly denoted by 2304(29) and its quaternionic representation will follow the group notation. We know that this is not a proper group notation; it should rather have a decomposition involving invariant subgroups. Since we denote each group by their elements the order with the conjugacy classes would be sufficient .

Maximal Subgroups of $Aut(F_4)$

We have three maximal subgroups of $Aut(F_4)$ of order 1152.

A : $W(F_4)$ of order 1152(25)

It is a subgroup of $O(4)$. We have discussed this group in details which was represented by the quaternions in (14) :

$$W(F_4) = [T, T] \oplus [T', T'] \oplus [T, T]^* \oplus [T', T']^*.$$

Note that the group $W(F_4)$ is invariant under the transformation $T \leftrightarrow T'$.

B : The group 1152(19)

It is a subgroup of $O(4)$ and its quaternionic structure can be written as follows

$$[T, T] \oplus [T', T'] \oplus [T, T']^* \oplus [T', T]^* \tag{34}$$

We know that the first two set of elements form a subgroup of order 576. One can show that the set of elements in (30) is closed by noting that

$$[T, T']^*[T, T']^* = [T', T]^*[T', T]^* = [T, T], \quad [T, T']^*[T', T]^* = [T', T']. \tag{35}$$

It is clear that it is a maximal subgroup of $Aut(F_4)$ and will be left invariant under the transformation $T \leftrightarrow T\prime$.

C : The group 1152(34)

This is the largest crystallographic group in 4-dimensions with proper rotations . That means it is a finite subgroup of $SO(4)$ .Naturally, it involves only non-star elements of $Aut(F_4)$

$$[T, T] \oplus [T', T'] \oplus [T, T'] \oplus [T', T]. \tag{36}$$

Its closure property is straightforward . Some of its subgroups of order 192 will be of our special interest for they appear as maximal subgroups in some of the finite subgroups of the Lie group G_2 [26]. It is also invariant under the transformation $T \leftrightarrow T'$.

Now we discuss, in turn, the maximal subgroups classified under the title A, B, C.

A.The Maximal Subgroups of $W(F_4)$

Its parabolic subgroups have been already discussed in section 4. Besides those groups there are two maximal subgroups of order 576 with the conjugacy classes 20 and 23. The group 576(23) is the extension of the Weyl group $W(SO(8))$ by a cyclic symmetry of the simple roots represented by imaginary quaternions. The group 576(20) is also a maximal subgroup of the groups 1152(19) and 1152(34).

A1. The group 576(23).

It is the extension of the of the group $W(SO(8))$ by a cyclic group of order 3 and its elements can be written as

$$[T, T] \oplus [T, T]^* \approx [T, T] \oplus [T', T']^*. \tag{37}$$

It can be shown that the group can be written as semi-direct product of the Weyl group of $SO(8)$ and the cyclic group Z_3,

$$W(SO(8)) : Z_3,$$

where the cyclic symmetry permutes the outer simple roots of $SO(8)$.

A2. The group 576(20).

It has the structure

$$[T, T] \oplus [T', T']. \tag{38}$$

It is also a maximal subgroup in the crystallographic subgroup of $SO(4)$ denoted by 1152(34) and the group 1152(19). No doubt that the elements in (38) closes under multiplication. We simply note the non-trivial case, namely,

$$[T\prime, T']^2 = [T, T].$$

B.The Maximal subgroups of the group 1152(19)

B1. The group 576(20).

This group is just discussed in A2.

B2. The group 192(17)

This group occurs also in the subgroup decomposition of the group 1152(34) and will be discussed under the subtitle C.

C. The Maximal subgroups of the group 1152(34)

C1. The group 576(29)

It has the structure

$$[T,T] \oplus [T,T'] \approx [T,T] \oplus [T',T] \tag{39}$$

Since the group 576(29) is an index 2 group in the group 1152(34) it should have two conjugates subgroups which is reflected in the isomorphism above.

C2. The group 384(31)

We have the following structure of the group

$$[V_0,T] \oplus [V_1,T] \oplus [V_0,T'] \oplus [V_1,T'] \tag{40}$$

This is certainly a maximal subgroup of the group 1152(34) because $V_0 \oplus V_1$ form a maximal subgroup of order 16 in the binary octahedral group $T \oplus T'$. It can be embedded in the group 1152(34) triply symmetric way by replacing V_1 by V_2 and V_3 in (40) in a similar manner where $W(SO(9))$ is embedded in $W(F_4)$.

C3. The group 288(24)

When we examine the parent group $[O,O]$ we know that the binary octahedral group O has many maximal subgroups one of which is the dicyclic group(binary dihedral group) of order 12. It can be generated by two elements $a = \frac{1}{2}(1-e_1-e_2-e_3)$ and $b = \frac{1}{\sqrt{2}}(e_1-e_2)$ where $a^6 = b^4 = 1$ satisfying the generation relation $ba^n\bar{b} = \bar{a}^n (n = 1,...,6)$. The group can be denoted by $2D_3$ where D_3 is the dihedral group of order 6. When $2D_3$ acts on the left and the binary octahedral group acts on the right we obtain the group 288(24) which reads in our notation

$$[2D_3,O] \tag{41}$$

We can further continue to determine the maximal subgroups of the groups discussed in the series A,B and C.

A1. The maximal subgroups of $W(SO(8)) : Z_3$

A1.1. The group 288(25) This is the group $[T,T]$ occurring in many groups discussed above.

A1.2. The group $192(13) \approx W(SO(8))$ It has been discussed before and shown to be the Weyl group of $SO(8)$

$$[V_0,V_0] \oplus [V_+,V_+] \oplus [V_-,V_-] \oplus [V_0,V_0]^* \oplus [V_+,V_+]^* \oplus [V_-,V_-]^* \tag{42}$$

which is invariant under the cyclic symmetry Z_3. The action of the group elements on the hyperoctahedra V_0, V_+, V_- are as follows:

i) $[V_0,V_0]$ leaves each hyperoctahedra invariant.

ii) $[V_+,V_+]$ permutes the three octahedra in the cyclic order and $[V_-,V_-]$ does the same in the reverse order.

iii) The element $[V_i,V_i]^* (i = 0,+,-)$ leaves the hyperoctahedron V_i invariant but interchanges the other two. These properties indicate that the $[V_0,V_0]$ form an invariant subgroup where the factor group is the symmetric group of order 6 $\frac{W(SO(8))}{[V_0,V_0]} \approx S_3$.

A1.3. The group 192(16) It can be represented in three equivalent ways and can be proven that they are the conjugate groups

i) $[V_0,V_0] \oplus [V_+,V_-] \oplus [V_-,V_+] \oplus [V_0,V_0]^* \oplus [V_+,V_-]^* \oplus [V_-,V_+]^*$

ii) $[V_0,V_0] \oplus [V_+,V_-] \oplus [V_-,V_+] \oplus [V_+,V_+]^* \oplus [V_-,V_0]^* \oplus [V_0,V_-]^*$

iii) $[V_0,V_0] \oplus [V_+,V_-] \oplus [V_-,V_+] \oplus [V_-,V_-]^* \oplus [V_+,V_0]^* \oplus [V_0,V_+]^*$

Interestingly enough that each of these conjugate groups leaves one of the hyperoctahedra invariant. One can easily show that the groups in(i),(ii) and (iii) leave V_0, V_+ and V_- invariant respectively. Embedding of the group 192(16) in the group $W(SO(8)) : Z_3$ follows the cyclic symmetry of quaternionic units e_1, e_2, e_3 .

B1. The Maximal subgroups of the group 576(20)

B1.1. The group 288(25) It has been discussed in A1.1.

B1.2. The group 288(24) This group was discussed in C3.

B1.3. The group $192'(13)$ It has the same order and the same number of conjugacy classes with $W(SO(8))$ but not isomorphic to it. It has the structure

$$[V_0, V_0] \oplus [V_+, V_+] \oplus [V_-, V_-] \oplus [V_1, V_1] \oplus [V_2, V_3] \oplus [V_3, V_2]. \tag{43}$$

An important difference is that $W(SO(8))$ is a subgroup of $O(4)$ whereas this group is a subgroup of $SO(4)$. The group $192'(13)$ has an index 6 in the group 1152(34). Its conjugate groups can be obtained by the conjugation of the element $[V_+, V_1]$ which permutes the 6 hyperoctahedra in the cyclic order $V_0 \rightarrow V_3 \rightarrow V_- \rightarrow V_1 \rightarrow V_+ \rightarrow V_2 \rightarrow V_0$. This would yield the 6 conjugate representations of (43).This group turns out to be a maximal subgroup of the finite subgroup of G_2 of order 1344 preserving the octonion algebra of the set $\pm e_i(i = 1, 2, ..., 7)$ [26].

C1.Maximal subgroups of the group 576(29)

All its maximal subgroups which have not been discussed so far also occur as the maximal subgroups of the group 384(31) and will be discussed below.

C2.Maximal subgroups of the group 384(31)

C2.1. The group 192(26) It has the structure

$$[V_0, T] \oplus [V_1, T] \tag{44}$$

C2.2. The group 192(23) It can be represented by

$$[V_0, T] \oplus [V_0, T'] \tag{45}$$

C2.3. The group 192(20) This group has an interesting structure which can be written as

$$[a, T] \oplus [b, T'] \tag{46}$$

where the set of elements of a is generated by $\frac{1}{\sqrt{2}}(1 + e_1)$ and the set $b = e_3 a$. The set $[a, T]$ forms an invariant subgroup of order 96. The set of elements a and b generate a dicyclic group of order 16 as we discussed before however as a and b are paired with different subsets of the binary octahedral group the dicyclic group is not a subgroup of the group 192(20). The set of elements of a and b are given by

$$a = \left\{\pm 1, \pm e_1, \frac{1}{\sqrt{2}}(\pm 1 \pm e_1)\right\}, \quad b = \left\{\pm e_2, \pm e_3, \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\right\} \tag{47}$$

C2.4. The group 192(17) It can be represented by

$$[V_0, T] \oplus [V_1, T'] \tag{48}$$

which is also a subgroup of the group 576(20). It is one of the maximal subgroup of the finite subgroup of the Lie group of G_2 order 12096 which leaves the quaternion decomposition of the octonionic root system of the exceptional Lie algebra E_7 [26] invariant.

CONCLUSION

The automorphism group $Aut(F_4)$ of the root system of the exceptional Lie algebra F_4 is the largest crystallographic group in 4-dimensions which has not been discussed in the literature using quaternions. This work not only relates this crystallographic group to the Coxeter- Dynkin diagram of F_4 but also discusses its relevance to other Lie algebraic

structures as well as to the 4-dimensional Euclidean geometry. We have discussed the decomposition of $Aut(F_4)$ down to the groups of order 192 and shown that a number of groups of order 192 have different structures related to different geometries. It is perhaps also interesting to continue the same decomposition to determine the groups acting in 3-dimensions . In this context we have discussed only the Weyl groups $W(S(7)) \approx W(SP(3))$.

We have noted that two groups of order 192, namely, the groups $192'(13)$ and $192(17)$ occur as maximal subgroups in the finite subgroups of the Lie group G_2. The group $192'(13)$ is a maximal subgroup of a group $Z_2^3 \cdot PSL_2(7)$ of order 1344 which is a finite subgroup of G_2 preserving the set of imaginary octonions $\pm e_i(i, 1, 2, \cdots, 7)$ [16]. Here $PSL_2(7)$ is the famous Klein's simple group of order 168 and $Z_2^3 = Z_2 \times Z_2 \times Z_2$ is the elementary abelian group of order 8. The group $192(17)$ is the maximal subgroup of the Chevalley group $G_2(2)$ of order 12096 which leaves the octonionic roots of the Lie algebra E_7 invariant .

The Weyl group $W(SO(8))$ which is the group $192(13)$ is also a maximal subgroup of a group $Z_2^3 : PSL_2(7)$ of order 1344 which is, in turn, a maximal subgroup of the simple group A_8, even permutations of 8 letters. The group A_8 is related to the Weyl group $W(E_7)$ through $W(SU(8)$ and is a maximal subgroup of the Chevalley group $SO_7(2)$ [27]. It is also interesting to note that some finite subgroups of $SO(4)$ also occur in the phase transitions of the liquid helium ^3He [28].

We believe that the group structures and their quaternionic representations will be useful in various fields of physics which may need the finite subgroups of $O(4)$.

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